A sentence is never more truthful than when it acknowledges itself a liar (finitely many times)*

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Broadly construed, and reductively stated, my thesis for this essay is:

Logic is physical.

A more accurate (but less punchy) statement would use softer language: that logic *ought* to be physical or (most accurately) that there is distinct explanatory benefit, under certain circumstances, to the encoding of physical considerations within logic and mathematics. This standpoint should be taken as one facet of a general view regarding the fundamental role of science as effectively communicating and collating lived (i.e. physical) experience and can be taken on par with such similar (or equivalent) statements as "information is physical" and "computation is physical". Of course, this thesis will likely be disputed by many, not in the least due to controversy over what "physical" actually entails. Seemingly, Rayo, at least implicitly, rejects this thesis since he takes logical possibility and physical possibility to be distinct concepts, as exemplified by the discussion regarding Thomson's Lamp [Chapter 3, Ray19]:

> "[E]ven if the relevant set-up is not physically possible, Thomson's Lamp can teach us something interesting about the limits of *logical* possibility." (emphasis in the original).

Despite this, many of the "paradoxes" presented in Rayo's book are presented with overtly physical, rather than purely formal, instantiations: bombs explode, lamps are turned on and off, people wear hats. As a result, I personally find it difficult to be persuaded by any such apparent paradox since, even if one accepts the assumption that logical and physical possibility are in fact distinct, then the physical reading of the paradox can be refuted on physical grounds and the logical reading is unclear since it is not presented using logical syntax (and hence difficult to identify the point of contention).

Before commencing properly with the main content, I would like to make it abundantly clear that my perspective is certainly **not** that a formal system or model of logic (or mathematics or computation) with fewer or no physical considerations has no scientific value. On the contrary, I think most of them *do* have significant worth in specific contexts and often-times physical considerations simply serve the purpose of muddying the analysis. As stated above, science is about effective communication; sometimes it is more effective to communicate using a language that has fewer physical considerations and sometimes it is not (and sometimes an apparent paradox is an indication of when the latter is the case). The explanatory power of the abstract (i.e. non-physical) notion of a Turing machine undoubtedly played a vital role in the development of computer science and computational complexity theory, but of course that doesn't stop my laptop running out of memory every two years (a far cry from the infinite tape I was promised). My main contentions, then, really are that one should at least be aware that every system of logic or mathematics is a convenient abstraction rather than some fundamental notion of "truth", that one should be willing to update the system when required, and that one would likely do well to update them with physically-motivated modifications. At least one of these (the middle one) seems to align well with Rayo's perspective, who opens Chapter 3 by listing "our reasoning isn't really valid" as one of the possible causes for the occurrence of a paradox.

In the remainder of this essay, I will argue the case for a single principle that I believe to be (almost) indisputably physical: finiteness. In slightly different terms, some notion of *resource-sensitivity* will be considered in our formal systems, which

*This is meant to be a play on the quote attributed to Mark Twain: *"A man is never more truthful than when he acknowledges himself a liar."*

I believe to be well-motivated by both quantum and classical physics. I focus on a single principle only due to time and space considerations (I only have a finite tape onto which I can write this essay and a finite time in which to do so) and I focus on *this* particular principle because I believe it to have stark consequences for the potential resolution of many "paradoxes". One more quote from Rayo [Conclusion of Chapter 1, Ray19]:

> "*Our discussion of infinity began with Hilbert's Hotel, which illustrates the fact that not all of our intuitions about finite cardinalities carry over to the infinite.*

My perspective can be interpreted as that many "paradoxes" arise from attempting to go the other way: carrying intuitions of infinite cardinalities to the finite (that is to say, the physical). I will proceed by discussing one example in detail, namely the Liar Paradox, as well as by informally discussing some general consequences. By way of a disclaimer, please note that, while I have tried to be as rigorous as possible where appropriate, most of this material is new to me, so I can make no guarantees, and much of what follows is informal. This essay should mostly be taken as an "in spirit" argument that logic (maths, computation, etc.) should be physical.

Let me also mention that many of the views presented in this essay seem to align closely with some of those presented in the Finitism and Ultrafinitism philosophies of mathematics [Wik22c; Wik21b]. I am certainly partial to these philosophies, however due to my ignorance of them, I am not (currently) willing to firmly call myself an Ultrafinitist or to claim that this essay is attempting to adhere to their doctrines.

Preliminaries

Because I would like to discuss the Liar's Paradox at some level of rigour, and moreover because I am aiming to argue that physical principles can and often should be implemented formally, it is prudent to start by introducing at least part of a formal system. The systems we will concern ourselves with are linear logic [Gir87], which already encodes notions of resource-sensitivity, and one of its variants [GSS92]. Before getting to (subsets of) the formal details of each of these logical systems, let us provide an illustration of the failure of classical (and in fact also intuitionistic) logic when it comes to notions of resource-sensitivity.

Consider a very basic truth table for two propositions A and B as well as the two formulae $A \implies B$ and $A \wedge B$:

A	B	$A \implies B$	$A \wedge B$
0	0	0	0
0	1	1	0
1	0	0	0
1	1	1	1

There are two conclusions that I would like to draw from the above truth table. Firstly, if A is true and $A \implies B$ is true, then B is true. This is the well-known rule of inference *modus ponens*. Secondly, if A is true and $A \implies B$ is true, then $A \wedge B$ is true. Does this mean that the two conclusions are equivalent? The following intuitive example, inspired by that given on the Wikipedia page for Linear Logic [Wik22d], suggests otherwise: Let A stand for "I have a dollar" and $A \implies B$ for "If I have a dollar, I use it to buy a banana". The first conclusion, B, is then that I have a banana. However, the second conclusion is that I have *both* my original dollar *and* the banana. Clearly, there is a difference between the two cases: in the latter, I have somehow gained resources for free (either by theft of the banana or by duplicating my dollar). Put another way, in the first case my resource A has been *used up* when passed to the conditional. It seems reasonable that these two cases be treated differently - this is what linear logic achieves, at least to some extent. In particular, linear logic includes a different style of conditional, denoted $\multimap$ below, which captures exactly this notion of consumption of an antecedent (A) to produce a consequent (B).

I would to rewrite the above two conclusions more rigorously, in order to introduce some notation of the sequent calculus. The first conclusion can be written as

$$\frac{\vdash A \qquad \vdash A \implies B}{\vdash B} \textit{ modus ponens}$$

where we are here using one-sided sequents in order to be more consistent with the formalism used below. The above should be read exactly the same as the written version in the preceding paragraph: if the two premises (above the line) hold, then the conclusion (below the line) also holds, based on the rule of inference indicated next to the line (*modus ponens*). The second conclusion is slightly more complicated, and actually uses a different set of premises (which is a point I return to below):

$$\dfrac{\dfrac{\vdash A \qquad \vdash A \qquad \vdash A \implies B}{\vdash A, B}\ \textit{modus ponens}}{\vdash A \wedge B}\ \wedge\text{-introduction}$$

Here $\wedge$-introduction is a typical rule of inference in many logics, but we skip any discussion of it since it is not so important for our present purposes. One could object to the addition A as a premise in this second case, however under the structural rule of inference called contraction, whose unfettered use is typically allowed in most classical (and intuitionistic) logics, we obtain the former case:

$$\dfrac{\vdash A \qquad \vdash A \qquad \vdash A \implies B}{\vdash A, \quad A \implies B}\ \text{contraction}$$

It is precisely this rule of inference which we will consider, and moreover restrict, through much of this essay due to its close connection to infinity: one could read the above proof tree in the reverse direction and think of the premise A being copied. Clearly, infinite repetition of this reverse reading then gives you infinite copies of A. Informally speaking, this contraction/copying is a statement regarding the equivalent "logical power" of a set of premises that contain multiplicities of premises with a set with no such multiplicities Thus, allowing unrestricted contraction is in some sense the same as equating access to infinite copies of a proposition with access to a single copy, which seems entirely unreasonable to me. We will see below that restricted use of contraction allows for a solution to the Liar Paradox.

It is time now to introduce Linear Logic, albeit not in its entirety; we will focus on the aspects most relevant for subsequent discussion. Much of what is presented follows that of [DM19; Wik22d]; one of the seminal works in Linear Logic is [Gir87]. Let us introduce the following symbols: the binary connectives $\otimes$, $\oplus$, & and $\parr$ sometimes called "tensor", "plus", "with" and "par" respectively, and the unary connectives ! ("of course") and ? ("why not") are called *exponentials* and will play a key role in our discussion. There also exist other symbols $1, 0, \top$ and $\perp$ which will essentially not explicitly feature in the remainder of this essay (althought it is worth noting that $A^{\perp}$ denotes negation). It is also useful to introduce linear implication, denoted '$\multimap$' and defined via $A \multimap B := A^{\perp} \parr B$, which is different to the implication '$\implies$' of classical logic - see the comments below.

To develop some intuition for the resource interpretation of linear logic, we can understand some of the above symbols as follows. The tensor $\otimes$ is the multiplicative conjunction, meaning that $A \otimes B$ represents the *simultaneous* occurrence of resources (i.e. parallel computation). In our banana example above, $A \otimes B$ then stands for having both the dollar and the banana at the same time (similar to the conjunction used above). The additive conjunction & indicates the *alternative* occurrence of resources. Extending our banana example to also include C (a carrot), which also costs one dollar to purchase, we can encode the ability to buy a banana and the ability to buy a carrot, but not both simultaneously, as $A \multimap B \& C$. It is not true that $A \multimap B \otimes C$ since $B \otimes C$ would cost two dollars, whereas we have access only to one. This also highlights the difference between $\multimap$ and $\implies$: the former is single-use only and can be considered as transforming the resources indicated by the antecedent to the resource indicated by the consequent. The notion of unlimited access to resources is encaptured by the exponentials ! and ? and in fact, the classical implication '$\implies$' can be recovered from linear implication via their use:

$$A \implies B \equiv\ !A \multimap B$$

This is the connection between the two implication as given by [GSS92]; the Wikipedia page state something differnt. Also according to of [GSS92], $!A$ can be read as 'A forever'.

As mentioned above, one key aspect under consideration in this essay is the restriction of structural rule of inference called contraction. It was presented above in the specific example, however its general form, in the case of classical logic, is

$$\dfrac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ \text{contraction}$$

where Γ and Δ are sets of formulae. In linear logic, contraction is valid only with the use of exponentials:

$$\frac{\Gamma, !A, !A \ \vdash \ \Delta}{\Gamma, !A \ \vdash \ \Delta}\,\text{contraction}$$

We introduce one other rule that we will need later, called initialisation or reflexivity:

$$\frac{}{A \ \vdash \ A}\,\text{init}$$

In the section on the Liar's Paradox, we will consider two types of restriction of the rule of contraction, namely via the rejection of the rule altogether in the classical case and via the use of bounded exponentials in the linear logic case. With regard to the second case, let us introduce one final piece of formalism.

There exist well-defined notions of boundedness in linear logic, such as that exemplified by [GSS92]. A full treatment of Bounded Linear Logic (BLL) is well beyond the scope of this essay (and my understanding), however I would like to mention some relevant aspects, namely that of the bounded exponential $!_x A$. Instead of 'A forever', the bounded exponential $!_x A$ can be interpreted as 'A until x' or 'A x times'. The corresponding rule of contraction is then given as

$$\frac{\Gamma, !_x A, !_y A \ \vdash \ \Delta}{\Gamma, !_{x+y} A \ \vdash \ \Delta}\,\text{contraction}$$

Finally, I would like to note that both classical logic and intuitionistic logic can be embedded into linear logic (in the unbounded case), which crucially relies on the presence of the (unbounded) exponentials ! and ?. The fragment of linear logic without exponentials, called *multiplicative additive linear logic* (MALL), is apparently strictly less expressive than classical logic [DM19]. One could then wonder about the expressiveness of BLL, to which the following theorem applies

Theorem (Thm 6.1, [GSS92])**.** *Every polynomial time computable function can be represented in BLL by a proof of* $\vdash \boldsymbol{N}_x^2 \multimap \boldsymbol{N}_{p(x)}^2$*, for some resource polynomial p.*

where "can be represented in BLL by a proof of" has a specific meaning within the formalism of BLL [Def 5.2, GSS92] (which needs to be checked in order to ensure that the computational complexity implications are to be understood correctly - it is not immediately obvious to me). I find the following quote from the introduction of [GSS92] a concise presentation of the intriguing perspective afforded by the above theorem:

"[C]ontrolling contraction is an indirect way of controlling time complexity."

I will return to this perspective in the next section.

Liar Paradox

We begin our discussion of the Liar Paradox by recalling its statement equivalent to that given by Rayo [Chapter 3, Ray19] but with notation that is closer to the treatment by Zardini [Zar11] which we will consider later:

($\mathfrak{l}$): The sentence $\mathfrak{l}$ is false.

To be more formal, we introduce the symbol T to denote the truth of a proposition P, that is, TP stands for "P is true" and is a valid proposition in our formal system. Using T, we can rewrite the Liar paradox as

$$\mathfrak{l} : \neg T\mathfrak{l}$$

For Zardini's argument, we also require the so-called correlation rules of T-introduction and T-elimination:

$$\begin{aligned} T\text{-introduction}: &\quad TP \text{ follows from } P \\ T\text{-elimination}: &\quad P \text{ follows from } TP \end{aligned}$$

For later reference, these are written in the sequent calculus as

$$\dfrac{}{P \vdash TP}\ T\text{-introduction}$$

and

$$\dfrac{}{TP \vdash P}\ T\text{-elimination}$$

I confess that I don't have a deep intuition for the origin of these rules - I believe they relate to considerations of how best to represent truth and falsity in a formal system, which seems to have a long and not entirely settled history (and hence is well beyond the scope of this essay). The level of understanding that I am operating at with these rules is simply that T-introduction states that if the proposition TP is true then the proposition P is also true, and T-elimination is the converse (if P is true then so is TP), which seems to be required for the ability to pass between the name of the liar sentence and its content.

The typical reasoning that supposedly produces a paradox from a liar sentence like $\mathfrak{l}$ above starts from the law of excluded middle and derives a violation of the law of non-contradiction for each sub-case Recall that the laws of excluded middle and non-contradiction for a proposition A are

$$\begin{aligned} \text{excluded middle:} &\quad \vdash A \vee \neg A \\ \text{non-contradiction:} &\quad \vdash \neg(A \wedge \neg A) \end{aligned}$$

In words, the paradoxical reasoning for the liar sentence $\mathfrak{l}$ starts from either the sentence being true, $T\mathfrak{l}$ or it being false $\not T\mathfrak{l}$. Then, for each case separately, arrives at the conclusion that both $T\mathfrak{l}$ and $\not T\mathfrak{l}$ are true (a rejection of the law of non-contradiction). We can write this formally for the case where $T\mathfrak{l}$ is true as follows (taken from [Zar11]):

$$\dfrac{\dfrac{\dfrac{}{T\mathfrak{l} \vdash T\mathfrak{l}}\ \text{init} \qquad \dfrac{}{T\mathfrak{l} \vdash \neg T\mathfrak{l}}\ T\text{-elimination}}{T\mathfrak{l}, T\mathfrak{l} \vdash T\mathfrak{l} \wedge \neg T\mathfrak{l}}\ \wedge\text{-introduction}}{T\mathfrak{l} \vdash T\mathfrak{l} \wedge \neg T\mathfrak{l}}\ \text{contraction}$$

where instead of writing the T-elimination as entailing $T\mathfrak{l} \vdash \mathfrak{l}$, we replace $\mathfrak{l}$ with its content $\neg T\mathfrak{l}$. For completeness, the sub-argument for the case where $\neg T\mathfrak{l}$ is true is:

$$\dfrac{\dfrac{\dfrac{}{\neg T\mathfrak{l} \vdash \neg T\mathfrak{l}}\ \text{init} \qquad \dfrac{}{\not T\mathfrak{l} \vdash T\mathfrak{l}}\ T\text{-introduction}}{\neg T\mathfrak{l}, \neg T\mathfrak{l} \vdash \neg T\mathfrak{l} \wedge T\mathfrak{l}}\ \wedge\text{-introduction}}{\neg T\mathfrak{l} \vdash \neg T\mathfrak{l} \wedge T\mathfrak{l}}\ \text{contraction}$$

It is clear that both sub-arguments make use of contraction in the final reduction step. If we disallow for contraction, then what is the state of affairs? Well, it is *not* the case that we have the truth of $T\mathfrak{l}$ entailing the rejection of the law of non-contradiction, but rather that the truth of *two* copies of $T\mathfrak{l}$ that provides the paradox (and similarly for the $\neg T\mathfrak{l}$ case). Thus, we are in fact blocked from starting with the law of excluded middle and ending up with an unavoidable rejection of the law of non-contradiction, since it is instead *two* copies of $T\mathfrak{l}$ or $\neg T\mathfrak{L}$ that entail the rejection and these two copies don't appear in the law of excluded middle. As far as I am concerned, this resolves the paradox.

Here I would like to summarise some of Zardini's comments regarding the resolution of paradoxes by restricting contradiction (see [Section 2.3, Zar11] which I highly recommend to the interested reader) since they seems to align closely with the quote given at the end of the preceding section. Zardini states that the failure to contract is an indication that the state-of-affairs is unstable. Informally, here unstable is taken to mean that some state-of-affairs is incompatible with their consequences, which is clearly the case of the sentence $\mathfrak{l}$, whose truth results in its falsity. Zardini then likens this to physical states at distinct points in time: one state-of-affairs, say $T\mathfrak{l}$, at time t_0 leads to another state-of-affairs, $\neg T\mathfrak{l}$, at t_1 and instability is the inability of the physical state at t_0 to coexist with that at t_1. Stability is the converse, where a physical state *is* able to coexist with its consequent physical state at a later time, and thus the original state is still available for further use. If a physical state is available for use for all times, then this corresponds well with the perspective of $!A$ as A forever. I would like to note that this perspective of (in)stability is shared, to some degree, by Rayo [Section 4, Ray13].

However, the removal of contraction altogether can be rather fatal: what remains is "so weak that one can hardly program more than programs permuting the components of a pair" [GSS92]. This is one of the motivations for merely restricting the applicability of contraction rather than removing it entirely I would love to give a formal treatment of the status of the Liar Paradox in this intermediate context, however my grasp of the formalism of BLL is not up to the task. What follows then is an informal discussion resting upon the perspective of instability and time complexity outlined previously.

For the intermediate case, let us take seriously the quote at the end of the previous section regarding contraction and time. One could recap the two extremes of allowing and disallowing contraction as follows. When contraction is not allowed, we are disallowing reuse of $\mathfrak{l}$ (or the stability of $\mathfrak{l}$) through time and hence no contradiction arises since $\mathfrak{l}$ as a label of the sentence and $\mathfrak{l}$ as the content of the sentence are distinct entities. When contraction *is* allowed, then so is unlimited reuse, and the label of the sentence and the content refer to the same entity and so an *unlimited* contradiction ensues. I emphasise the word 'unlimited' here since I want to take the perspective that allowing bounded use of contraction, such as in the intermediate case of the contraction rule using $!_x$ and $!_y$ that we are considering, entails a *bounded* contradiction. For example, imagine that I can store $\mathfrak{l}$ stably in the memory of my laptop for up to two years at which point the deterioration of my laptop hardware induces instability in its memory. At the two year mark, we no longer have the ability to access $\mathfrak{l}$. In this case, we have (seemingly) experienced a paradox for the first two years, but then when my laptop's memory deteriorates, the paradox is resolved. I would argue that this in fact resolves the paradox *completely* since one merely needs to wait for the (inevitable) onset of instability to obtain the resolution. Said another way, the Liar Paradox (and I believe many other paradoxes) *only* exists in the presence of infinity, which is to say that it doesn't exist at all (since infinity does not either).

I would like to make some comments regarding the use of the term 'bounded' as opposed to 'finite'. My limited (bounded, finite, etc.) understanding of BLL is that the exponentials ! and ? are taken to be restricted by some resource polynomial $p(x)$, which of course does not immediately mean that they are restricted by some absolute finite upper bound. After all, one of the motivations for BLL is to encapture the notion of polynomial time computational complexity within a formal system and polynomial time computational complexity is not usually taken as a statement about finitary computation but about the growth rate of complexity with problem size. My opinion really is that such an upper bound does in fact exist, in an ontological or perhaps just a pragmatic sense. For example, perhaps we should take ! and ? to be restricted by some $\hat{x}$ where $\hat{x}$ is, say, any transcomputational number [Wik21a] or $A(5,5)$ [Kor03] where A is the Ackermann function [Wik22a]. I consider boundedness in this context as being a useful way of treating such an absolute bound under some amount of uncertainty of what specific value it should take (but perhaps with knowledge of some reasonable relationship between resources required or available to problem size), just as infinity can perhaps be considered to be a convenient way of course-graining even further (i.e. without knowledge of the relation to problem size).

Other Consequences and Comments

In this section, we will briefly consider some further consequences of the main perspective of this essay. These discussions will continue to be informal.

Gödel:

Recall the statement of Gödel's Incompleteness Theorem as given by Rayo [Chapter 10, Ray19]:

> *No axiomatization of elementary arithmetic in a rich-enough language can be both consistent and complete.*

Recall also that consistency refers to the fact that property of the axioms of a formal system such that no statement exists where both the axiom and its negation are provable from the axioms (essentially related to the law of non-contradiction in some sense) and that completeness refers to the property that for any statement in the formal system either it or its negation is provable from the axioms (essentially related to the law of excluded middle). We have already seen how the restriction of contraction relates to these laws in the discussion of the Liar Paradox, and we briefly consider here the connections to completeness and consistency. My understanding of the (lack of) expressibility of linear logic without exponentials (MALL), which in particular means also no contraction, is such that we don't fulfil the conditions of the theorem (but I don't have an explicit reference for this and I'm also not sure how MALL and classical logic without contraction relate). It also seems apparent that, if we allow for contraction but take a hard finitary stance and enforce an absolute bound on our formal system, then we also would not satisfy the conditions of the above theorem. Thus, at least in principle, our two formal systems that restrict contraction could possibly be both consistent and complete.

Partly for my own future reference and for the interested reader, the following literature seems to be relevant to discussions of completeness and consistency in the present context. In the Wikipedia article on Ultrafinitism [Wik21b], Alexander Esenin-Volpin [Wik22b] is mentioned as having initiated a program for proving the consistency of Zermelo-Fraenkel set theory using ultrafinite mathematics (the following are referenced as specific work in this direction: [Yes70; Yes81]). In [MC06] (which appears to be a preprint associated to the chapter entitled "A very short history of ultrafinitism" in [KK11]), Parikh is cited as having introduced a version of Peano arithmetic which includes a predicate encoding 'feasibility' and for having shown that this modified arithmetic is feasibly consistent for a specific finite bound [Par71] (here feasibility restricts the length of a proof of a statement, where length is the number of symbols used). Please note that I have not read these works in any great detail and am merely reporting their potential relevance.

I would also be interested in understanding better the applicability of Gödel's theorem for BLL where exponentials are bounded polynomially rather than by a fixed constant. After a very superficial search, the closest connection I could find between BLL and Gödel's theorem is contained in the thesis of Samuel Buss ([Bus85], see especially Chapter 7), however according to the introduction of the [GSS92], there are difference between BLL as presented there and the bounded arithmetic of Buss (differences that I do not currently understand).

Computation/Computability:

Notions of computability and complexity have been mentioned a few times throughout this essay and I don't have much more to say. I would like to reiterate that considerations of polynomial time computation are listed as some of the main motivations for the development of BLL and Buss's bounded arithmetic. Furthermore, since various connections between linear logic and quantum computation/quantum logic already exist (see e.g. the nlab entry on quantum logic [nLa22] and the references therein, including [BS10; Yet90; Pra93; AD06]), I would be interested to investigate what can (or has) been said with regard to the bounded case, such as whether the complexity class BQP can (or has) be interpreted in some formal system similar to how BLL captures P (I'm not sure if BPP has been treated in such a way either - these considerations were beyond my scope for this essay). In particular, I wonder if the notion of 'stability' associated to contraction as outlined in the discussion of the Liar Paradox could be related to the coherence of a quantum system: for example, perhaps each interaction of a system (our premise $!_x A$) with an environment can be considered (in some loose sense) as the 'use' of one of the As, and hence after x interactions, we have reached our allowed resource bound and the system becomes decoherent thereafter. These are clearly very under-developed and likely entirely nonsensical ideas, so please treat them as such.

Finally, again mostly for my own later referral, I wanted to mention works such as [Llo02; Kor03], which consider some aspects of finiteness (of time, space, energy, etc.) for computation despite being relatively light (in my opinion) on the broader consequences thereof for mathematics, logic, etc.

Limitations and Disadvantages:

One can likely state many limitations and disadvantages of the finitary viewpoint expounded in this essay. I think the strongest one is that it makes various aspects of the analysis more complicated, in both technical and conceptual aspects. I also think that this is entirely to be expected: the closer a model is to the complex thing it is modelling, of course the model will be more complicated. One then has to weigh up whether the disadvantages of this viewpoint, or a formal system motivated by it, are greater than the disadvantages of an alternative. Certainly, the point of view investigated here is not optimal or appropriate in all cases - no single point of view is - but it does seem to be appropriate in at least *some* cases, such as in providing a resolution to various paradoxes.

Conclusion

Mathematics and logic, just like all other sciences, is a means of communicating a certain type of lived experiences. And, again just like all other sciences, when that means of communication is deemed inadequate for a specific instance, such as when a paradox seemingly arises, then this is an indication that the language may need refining (quite literally in our case). Here, we have argued that, in many cases, a refinement based on physically-motivated principles adequately resolves the problem for the instance in question (possibly at the cost of introducing inadequacies elsewhere). Specifically, we considered how the unquestioning acceptance of the (non-physical) notion of infinity in certain logics can produce paradoxes such as the Liar Paradox, and how a more nuanced approach that restricts or removes infinity also restricts or removes such paradoxes.

We are finite beings living in a finite universe, with finite access to finite physical systems and finite amounts of energy for a finite amount of time. No doubt, infinity can be an efficient way of conceptualising large quantities within finite storage space (our brains), but perhaps it's not so efficient to occupy too much of that storage space considering whether a sentence is lying or not. There are (infinitely) many other interesting problems to consider.

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