Skolem's Paradox

Tobias Reinhart

Skolem's paradox [1] is the seeming contradiction between the fact that on the one hand, ZFC set theory proves the existence of uncountable sets while on the other hand, according to the Löwenheim-Skolem theorem, there exist countable models of ZFC. A model of ZFC, $\mathcal{M}$, consists of a set M and a binary relation $\in^{\mathcal{M}}$ on M such that all theorems of ZFC hold when interpreted in $\mathcal{M}$, where interpreting a term of ZFC in $\mathcal{M}$ amounts to replacing $\in$ with $\in^{\mathcal{M}}$ and restricting all quantifications $\forall, \exists$ to M instead of quantifying over all sets.

Skolem's paradox now follows from the existence of transitive, countable models of ZFC. In such models, first, $\in^{\mathcal{M}}$ agrees with $\in$ and second, M and all its elements are countable sets [2], i.e. can be enumerated by the natural numbers. As, however, M realizes all theorems of ZFC, in particular, it realizes the theorem "there exist uncountable sets". Thus, there exist sets in M that are countable, when seen from outside of $\mathcal{M}$ but uncountable, when seen from inside of $\mathcal{M}$.

Although seemingly a contradiction, Skolem's paradox does not reveal mathematical flaws of set theory, it rather highlights limitations of our naive reasoning about it. While the existence of uncountable sets can be derived from the ZFC axioms and thus holds in every model, i.e. can be seen as absolute, other notions such as the specific cardinality of sets are not absolute. Claiming that a certain set has a certain cardinality only makes sense relative to a given model. In this text, I will argue in more depth, how, in this light, we can make sense out of Skolem's paradox.

1 ZFC as a first-order theory

In order to fully understand Skolem's paradox, one first needs to understand ZFC as a first-order theory. In particular, it is important to note that, in this context, ZFC is purely syntactical. Writing $a \in A$, we do not need to interpret this as "a is an element of A", nor do we need to interpret a and A as sets. At this point a and A are simply variables and $a \in A$ is a well-formed term of ZFC.

Definition 1. *The language of ZFC, $\mathcal{L}_{\text{ZFC}}$ consists of variables $A, a, B, b, \ldots$, logical connectives $\wedge, \vee, \neg, \implies, \iff, \ldots$, quantifiers $\forall, \exists$ and two binary predicates $=, \in$.*

This means that any well-formed[1] string of these symbols is a valid formula

[1] Well-formed according to the usual rules of these symbols.

in ZFC. A formula is called a sentence if it contains no free variables, i.e. all variables are bound by quantifiers.

A theory consists of a language $\mathcal{L}$ together with a collection of sentences of $\mathcal{L}$, called axioms. ZFC is defined by the following axioms.

Definition 2 (ZFC)**.** *ZFC is the theory over $\mathcal{L}_{\text{ZFC}}$ that is defined by the well known axioms (see for instance [3]).*

For this text, it is less important how these axioms precisely look like. It is however important that each axiom of ZFC simply is a sentence of $\mathcal{L}_{\text{ZFC}}$.

In order to work with ZFC, we additionally have to prescribe a mechanism of how we can derive new sentences from these axioms, i.e. of how we can prove theorems.

Definition 3 (see [4])**.** *Let ϕ be a sentence in $\mathcal{L}_{\text{ZFC}}$. A proof of ϕ in ZFC is a sequence of formulas of ZFC $\psi_1, ..., \psi_n$ such that*

1. *the first formula ψ_1 is an axiom of ZFC*
2. *the last formula ψ_n equals ϕ*
3. *ψ_i is either an axiom or follows from $\psi_1, ..., \psi_{i-1}$ by logical inference rules (modus ponens, ...)*[2]

If there exists a proof for ϕ in ZFC we write $\text{ZFC} \vdash \phi$ *and say ZFC proves ϕ or ϕ is a theorem in ZFC.*

Note that at this point, everything is still purely syntactical. There is no need to interpret meaning into axioms of ZFC. At this point, ZFC, together with the proof system of choice, simply is a formal language with a distinguished collection of sentences (the axioms) and a mechanism of constructing new sentences from the axioms (the theorems).

2 Models of ZFC

In contrast to the purely syntactical treatment in the previous section, in mathematical practice, writing $a \in A$ we do interpret a and A as sets and $a \in A$ as "a is an element of A". More precisely, usually we do not think of ZFC as a first-order theory but rather think of a model of ZFC, i.e. we imagine a mathematical universe of sets M that is equipped with a binary element relation $\in$. This point of view is made precise in the context of model theory. In particular, in model theory, there is a clear distinction between the purely syntactical, axiomatic system and models of this system. Roughly speaking, a model of an axiomatic system is a collection of sets and relations between these sets such that all axioms are satisfied when mapping variables $A, a, ...$ to sets and predicates $\in, ...$ to relations. Before making this precise, we have to define the weaker notion of a structure for a formal system.

[2] More abstractly, several choices of these inference rules are possible. To specify a proof system one needs to specify the allowed inference rules.

Definition 4 (see [4])**.** *An $\mathcal{L}_{\text{ZFC}}$-structure $\mathcal{M}$ consists of a set M and a binary relation $\in^{\mathcal{M}}$ on M such that for each variable a of $\mathcal{L}_{\text{ZFC}}$ there is a corresponding element $a^{\mathcal{M}}$ of $\mathcal{M}$.*

We call M the domain or universe of $\mathcal{M}$ and $\in^{\mathcal{M}}$ its element relation. Note that an $\mathcal{L}_{\text{ZFC}}$-structure gives meaning to ZFC, within $\mathcal{M}$ variables $a, b, c, ...$ now correspond to actual set $a^{\mathcal{M}}, b^{\mathcal{M}}, c^{\mathcal{M}}, ...$ and $\in^{\mathcal{M}}$ is an actual relation among those sets, not just a predicate symbol. Let ϕ be a sentence of ZFC. We can inductively define $\mathcal{M} \models \phi$ [4]. Roughly speaking, $\mathcal{M} \models \phi$ if ϕ evaluates to true when

1. replacing all occurrences of $\in$ in ϕ with $\in^{\mathcal{M}}$
2. replacing all occurrences of variables $a, b, c, ...$ in ϕ with their corresponding sets $a^{\mathcal{M}}, b^{\mathcal{M}}, c^{\mathcal{M}}$
3. replacing all quantifications $\forall a, \exists b, ...$ with $\forall a \in M, \exists b \in M, ...$

We read $\mathcal{M} \models \phi$ as $\mathcal{M}$ satisfies ϕ or ϕ is true in $\mathcal{M}$.

Definition 5. *An $\mathcal{L}_{\text{ZFC}}$-structure $\mathcal{M}$ is a model for ZFC if for every axiom of ZFC ϕ, $\mathcal{M} \models \phi$.*

Note that there might exist multiple models of ZFC that differ in their universes as well as in their element relation. In particular, given a sentence ϕ of ZFC, there might exist models that satisfy ϕ and such that do not satisfy ϕ. If a sentence ϕ is satisfied in every model of ZFC we write ZFC $\models \phi$. Gödel's completeness theorem states that this is the case if and only if ϕ is a theorem of ZFC, i.e.

$$\text{ZFC} \models \phi \quad \text{if and only if} \quad \text{ZFC} \vdash \phi \tag{1}$$

The situation hence is the following: ZFC is a purely syntactical first-order theory that together with a proof systems allows us to derive theorems. In practice, we mainly study[3] models of ZFC. These models allow us to talk about actual sets and relations between them. Moreover, within each model, certain statements about sets will be true. Precisely those statements that are theorems of ZFC are true in every model. Conversely, if a statement is true in one model but its negation is true in some other model, this statement is independent of ZFC, i.e. neither the statement, nor its negation is a theorem in ZFC.

Now coming back to the introduction of this section, when we think of ZFC we think of a specific model $\mathcal{U}$ of it. This model is special as on the one hand, its universe U should contain all sets[4] that can be constructed from ZFC while on the other hand its element relation $\in^{\mathcal{U}}$ should agree with the usual $\in$ of ZFC. Note that, while $\in$ is a binary predicate, i.e. can be understood as purely syntactical symbol, $\in^{\mathcal{U}}$ is an actual relation on U, i.e. can be understood as a subset of U^2. Agreement of $\in^{\mathcal{U}}$ with $\in$ should hence be understood in the sense that $a \in b$ can be proven from ZFC if and only if $a \in^{\mathcal{U}} b$ according to

[3] or think of

[4] Or at least all sets of interest.

$\mathcal{U}$. One possible such model is obtained by taking U to be the von Neumann universe [5]. The construction of the van Neumann universe is similar to the construction of ordinal numbers in [6]. The problem, however, is that that U is not a set itself but a proper class. One can nevertheless interpret $\mathcal{U}$ as the set-theoretic universe, by relaxing the notion of a model of ZFC to allow for class-models instead of set-models. Now the crucial point is that whenever we make statements about properties of certain sets it usually is understood w.r.t. this model $\mathcal{U}$. In particular, certain statements might hold in this model but fail to hold in other models of ZFC[5]. Such statements are then not absolute, in the sense that they can be proven from the axioms of ZFC but relative to the model that we chose to work with.

An interesting class of models of ZFC can be obtained by restricting the set theoretic universe U to a subclass of it that is small enough to form a set. On this restricted universe one then uses the restricted version of $\in^{\mathcal{U}}$ as element relation.

Definition 6 (see [7])**.** *A model $\mathcal{M}$ is called standard if its element relation $\in^{\mathcal{M}}$ is the restriction of $\in^{\mathcal{U}}$ to M, i.e. if its element relation agrees with $\in$. $\mathcal{M}$ is called transitive if it is standard and M is a transitive set, that is every element of M is also a subset of M*[6]*.*

Note that standard model of ZFC, still have an element relation that agrees with $\in$[7]. They, however, have a restricted universe, that only is a subset of the proper class U that contains all sets and quantification is understood w.r.t. this restricted universe.

Note further that ZFC does not prove the existence of transitive models of ZFC. It does not even prove the existence of models of it in general, as if a theory has a model it is consistent [4] and Gödel's incompleteness theorems states that ZFC cannot prove its own consistency. For the following, we will nevertheless assume the existence of a transitive model of ZFC. This transitive model sets the stage for Skolem's paradox, as we can apply the transitive submodel theorem, a stronger version of the downward Löwenheim-Skolem theorem, to it.

Theorem 1 (see [8])**.** *If ZFC has a transitive model, then it has a transitive model with countable universe.*

3 Skolem's paradox

We are now in a position where we can discuss Skolem's paradox on a formal level. Assuming the existence of a transitive model of ZFC, according to the transitive submodel theorem, there exists a transitive model $\mathcal{M}$ with countable universe M. Note that stating that M is countable has to be understood w.r.t.

[5]And this isprecisely what happens in Skolem's paradox.

[6]For instance $\{\emptyset, \{\emptyset\}\}$ is transitive. Moreover every ordinal number is transitive and even every element of an ordinal number is transitive.

[7]People often state that standard models of ZFC use the "real" element relation.

$\mathcal{U}$. More precisely, M and $\mathbb{N}$ are contained[8] in U and "M is countable" means that in $\mathcal{U}$ there exists a injective function from M to $\mathbb{N}$, i.e. an enumeration of M. This, in particular, implies that every element of M is countable: As M is transitive, every element m of it is also a subset of it. Restricting the enumeration to m (as a subset of M) hence provides an enumeration of m (as an element).

Next, recall that as $\mathcal{M}$ is a model of ZFC, it satisfies at least those sentences that can be proven in ZFC, i.e. if $ZFC \vdash \phi$ then $\mathcal{M} \models \phi$. In particular, the existence of the set of natural numbers can be proven from ZFC. More precisely, there exists a formula in $\mathcal{L}_{\text{ZFC}}$, $\phi(N)$ with one free variable, N, that states "N are the natural numbers" and $\exists! N : \phi(N)$ is a theorem of ZFC [9].

Moreover, Cantor's theorem, "no set has the same cardinality as its powerset", is also a theorem of ZFC. Given a set S, the existence of its powerset follows from an axiom of ZFC. Thus, it follows that for each set S there exists a set P (one choice for P is $\mathcal{P}(S)$) with strictly bigger cardinality, or more formally, the following is a theorem in ZFC

$$\forall S : \exists P : \forall f : \text{"}f \text{ is a function from } S \text{ to } P\text{"} \implies \text{"}f \text{ is not surjective"} \quad (2)$$

Combining this with the theorem that guarantees the existence of the set of natural numbers, we get the following new theorem in ZFC:

$$\begin{aligned}\psi := \exists! N : \exists P : \phi(N) \\ \wedge (\forall f : \text{"}f \text{ is a function from } N \text{ to } P\text{"} \implies \text{"}f \text{ is not surjective)"}\end{aligned} \quad (3)$$

In words, this states that ZFC proves the existence of the set of natural numbers together with a set P that cannot be enumerated by these natural numbers, i.e. that is uncountable w.r.t. these natural numbers. Now this theorem is satisfied by $\mathcal{M}$, i.e. $\mathcal{M} \models \psi$.

As $\mathcal{M}$ is a standard model of ZFC, interpreting ψ w.r.t. $\mathcal{M}$ amounts to replacing all unrestricted quantifications $\forall a, \exists b, ...$ with restricted quantifications $\forall a \in M, \exists b \in M, ...$. Thus, in words, the formula that we arrive at states the following: "In M there exists a set of natural numbers N and a set P such that in M there exists no enumeration of P by elements of N. This is Skolem's paradox.

Definition 7 (Skolem's paradox)**.** *Assume the existence of a transitive model of ZFC, then there exists a transitive model $\mathcal{M}$ such that*

1. *M and all its elements are countable sets (w.r.t. the ambient model $\mathcal{U}$),*
2. *in M there exists a set P that w.r.t. $\mathcal{M}$ (and w.r.t. the natural numbers in $\mathcal{M}$) is uncountable.*

If one assumes that a given set is either countable or uncountable in an absolute way, this is clearly a paradox. How can P be countable, when seen

[8] Here $\mathbb{N}$ refers to the usual, set-theoretic construction of the natural numbers.

from the outside of $\mathcal{M}$, i.e. w.r.t. $\mathcal{U}$ but uncountable when seen from the inside of $\mathcal{M}$. Interpreting Skolem's paradox from a more technical, model theoretic point of view it is however not so surprising. Clearly, there exist statements that depend on the model that one uses to interpret them, i.e. that are relative to the model and not absolute. This is the very idea that allows one to prove independence of a certain sentence from a given axiomatic system in a model theoretic way, by constructing both a model that satisfies that sentence and a model that satisfies its negation. From this point of view, Skolem's paradox simply reveals that a set cannot be uncountable in an absolute way but only relative to a given model.

Moreover, from this model theoretic point of view it is even clear how P can be countable w.r.t. $\mathcal{U}$ but uncountable w.r.t. $\mathcal{M}$. To that end, note that it can be shown that the set of natural numbers, characterized as the unique set N that satisfies $\phi(N)$ is the same in all transitive models of ZFC[10]. This means that, restricting ourselves to transitive models, we can safely talk about the set of natural numbers $\mathbb{N}$. Next, recall that $\mathcal{M}$ and $\mathcal{U}$, when restricted to sets contained in M have the same element relation. Hence, the only difference that occurs when interpreting formulas w.r.t. $\mathcal{M}$ instead of $\mathcal{U}$ is that quantifiers are bounded to M. Stating that P is countable w.r.t. $\mathcal{U}$ thus means that in U there exists an enumeration of P by elements of $\mathbb{N}$, i.e. an injective function[9] from $\mathbb{N}$ to P. Stating that P is countable w.r.t. $\mathcal{M}$ then means that there exists such an enumeration in M. Hence, if the enumeration that witnesses that P is countable is an element of U but not an element M, P is countable w.r.t. $\mathcal{U}$ but uncountable w.r.t. M, which perfectly explains Skolem's paradox.

In total, from a model theoretic point of view Skolem's paradox does not seem so paradoxical anymore. It rather shows us that certain properties of sets, that we naively interpret in an absolute way are actually relative to the model in which we interpret them.

References

[1] T. Skolem, *Einige bemerkungen zur axiomatischen begründung der mengenlehre*, 1922.

[2] T. Bays, in *Philosophy of logic*, edited by D. Jacquette (North Holland, 2006), pp. 615–648.

[3] T. Jech, *Set theory: the third millennium edition* (Springer, 2003).

[4] D. Marker, *Model theory : an introduction*, Graduate Texts in Mathematics (Springer New York, 2002).

[5] Wikipedia contributors, *Von neumann universe — Wikipedia, the free encyclopedia*, `https://en.wikipedia.org/w/index.php?title=Von_Neumann_universe&oldid=1068449716`, [Online; accessed 11-July-2022], 2022.

[9] Note that in ZFC a function $f : A \to B$ simple is a subset of $A \times B$ that satisfies certain properties (think of the graph of the function).

[6] A. Rayo, *On the brink of paradox: highlights from the intersection of philosophy and mathematics* (MIT Press, 2019).

[7] G. Takeuti and W. Zaring, *Introduction to axiomatic set theory*, Graduate texts in mathematics (Springer-Verlag, 1982).

[8] T. Bays, in *The Stanford encyclopedia of philosophy*, edited by E. N. Zalta, Winter 2014 (Metaphysics Research Lab, Stanford University, 2014).

[9] Wikipedia contributors, *Axiom of infinity — Wikipedia, the free encyclopedia*, `https://en.wikipedia.org/w/index.php?title=Axiom_of_infinity&oldid=1097352538`, [Online; accessed 11-July-2022], 2022.

[10] Rowan Jacobs, *Forcing*, `http://www.math.uchicago.edu/~may/VIGRE/VIGRE2011/REUPapers/Jacobs.pdf`, [Online; accessed 11-July-2022].